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Non-Static Local String in Higher-Dimensional Gravity

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We analyze the space-time structure of local gauge string with a phenomenological energy–momentum tensor, as prescribed by Vilenkin, in an arbitrary number of space-time dimensions with a non-zero cosmological constant Λ. A set of solutions of the full non-linear Einstein's equations for the interior region of such a string is presented.

KEY WORDS: Non static local string; higher dimension; Exact solutions.

Finding a theory that unifies gravity with other forces in nature has been an elusive goal for theoretical physicists. At first, Kaluza–Klein (K–K) showed how gravity and electro magnetism can be unified from Einstein's field equations generalized to five dimensions (Kaluza, 1921; Klein, 1926; Weinberg, 1986). After that, higher-dimensional cosmological models have been quite often present in scientific research. In recent years, there has been renewed interest in "brane-world" models in which the universe is represented by a $(3 + 1)$-dimensional subspace (3-brane) embedded in a higher-dimensional (bulk) space-time (Arkani-Hamed *et al.*, 1998). The idea of brane world may resolve the challenging problem in theoretical physics, namely the unification of all forces and particles in nature. According to Randall and Sundrum (1999a,b), it is possible to have a single massless bound state confined to a domain wall or 3-brane in five-dimensional non-factorizable geometries. The brane is pictured on a domain wall propagating in a five-dimensional bulk space-time. They have shown that this bound state corresponds to the zero mode of Kaluza–Klein dimensional reduction and is related to four-dimensional gravity Randall and Sundrum (1999a,b). Recently Cohen and Kaplan (1999) have shown that the global string has a curvature singularity at a finite distance from the string core. They argued that the singularity can provide

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an effective compactification of extra dimension. Gregory (2000) has shown that a non-singular global string exists in presence of negative cosmological constant. In her solution, the extra dimension are infinite and strongly warped as in Randall and Sundrum model. Olasagasti and Vilenkin (2000) obtained solutions of Einstein equations for global defects in higher-dimensional space-time with a non-zero cosmological constant.

In this report we explore solutions of Einstein equations for local string in a higher-dimensional space-time with a non-zero cosmological constant.

We assume an infinite long straight string, characterized by the energy momentum tensor components $T_t^t = T_{zi}^{zi} \neq 0$ and all other T_i^k's are zero (Vilenkin, 1985). Because of the relevance of the brane scenario in the context of cosmic string, it becomes necessary to investigate whether a local gauge string can give rise to essential solutions of gravitational field equations in a higher-dimensional space-time with a non-zero cosmological constant.

We adopt the following ansatz for the metric

$$ds^2 = e^{2A(r)}\left[dt^2 - e^{2b(t)} dz_{\mathrm{i}}^2\right] - dr^2 - C^2(r)\, d\theta^2 \tag{1}$$

where $i = 1, 2, \ldots p$ are the spatial coordinates in our lower-dimensional space-time.

The local string is characterized by an energy density and stresses along the symmetry axes given by

$$T_t^t = T_{zi}^{zi} = F(r,t) \tag{2}$$

$$\text{and} \quad T_r^r = T_\theta^\theta = 0 \tag{3}$$

The system of equations for a local string is

$$\left(\frac{C^{11}}{C}\right) + p\left[A^{11} + (A^1)^2 + \left(\frac{A^1 C^1}{C}\right)\right] + \frac{1}{2}p(p-1)[(A^1)^2 - (b^{\cdot})^2 e^{-2A}] = 8\pi G F(r,t) e^{2A} - \Lambda \tag{4}$$

$$p[b^{\cdot\cdot} + (b^{\cdot})^2 e^{-2A}] - p(A^1)^2 - (p+1)\left(\frac{A^1 C^1}{C}\right) - \frac{1}{2}p(p-1)[(A^1)^2 - (b^{\cdot})^2 e^{-2A}] = -\Lambda \tag{5}$$

$$p[b^{\cdot\cdot} + (b^{\cdot})^2 e^{-2A}] - (2p+1)(A^1)^2 - (p+1)A^{11} - \frac{1}{2}p(p-1)[(A^1)^2 - (b^{\cdot})^2 e^{-2A}] = -\Lambda \tag{6}$$

['^{1}' and '$^{\cdot}$' represent differentiations w.r.t 'r' and t, respectively.]

From Eqs. (5) and (6) one can write

$$\left(\frac{C^1}{C}\right) = \left(\frac{A^{11}}{A^1}\right) + A^1 \tag{7}$$

Eq. (7) readily integrates to yield

$$C = (e^A)^1 \tag{8}$$

where the constant of integration has been absorbed by rescaling the radial coordinate without any loss of generality.

Now from eq. (5), by using eq. (8), we get

$$p\left[b^{\cdot\cdot} + \frac{1}{2}(p+1)(b^{\cdot})^2\right]$$
$$= e^{2A}\left[\frac{1}{2}p(p+1)(A^1)^2 + (p+1)\{(A^1)^2 + A^{11}\} - \Lambda]\right] = b_0 \tag{9}$$

where b_0 is the separation constant.

Now one can separate time and space parts as

$$\left[b^{\cdot\cdot} + \frac{1}{2}(p+1)(b^{\cdot})^2\right] = \left(\frac{b_0}{p}\right) \tag{10}$$

$$A^{11} + a(A^1)^2 = c + de^{-2A} \tag{11}$$

where $a = \frac{1}{2}(p+2)$, $c = [\Lambda/(p+1)]$ and $d = [b_0/(p+1)]$ In what follows we shall try to solve the system of equations for $b_0 = 0$ and for $b_0 \neq 0$.

1. CASE – I : $\mathbf{b_0 = 0}$

In this case from Eq. (10), we can solve b which becomes

$$e^b = t^{[2/(p+1)]} \tag{12}$$

where the constant of integration has been absorbed by rescaling the time coordinate without any loss of generality.

Now one can integrate Eq. (11) to yield

$$A = \left(\frac{1}{a}\right) \ln[\cosh(aH)r] \tag{13}$$

where $H^2 = (c/a)$ using Eq. (13), one gets from Eq. (8) as

$$C = H[\sinh(aH)r][\cosh(aH)r]^{[(1-a)/a]} \tag{14}$$

The string energy energy density $F(r, t)$ can be found from Eq. (4) which becomes

$$8\pi GF(r,t) = \Big[\cosh(aH)r]^{(-2/a)}[\{\Lambda + 3aH^2 - 2a^2H^2 + 2apH^2\} - \left\{\frac{2p(p-1)}{(p+1)t^2}\right\}\{\cosh(aH)r\}^{(-2/a)} + \{\tanh(aH)r\}^2 \times \left\{H^2 - 3aH^2 + 2a^2H^2 - 2apH^2 + \left(\frac{3}{2}\right)pH^2 + \frac{1}{2}p^2H^2\right\}\Big] \quad (15)$$

Finally the line element becomes

$$ds^2 = [\cosh(aH)r]^{(2/a)}\big[dt^2 - t^{[4/(p+1)]}dz_i^2\big] - dr^2 - H^2[\sinh(aH)r]^2[\cosh(aH)r]^{[2(1-a)/a]}\, d\theta^2 \quad (16)$$

One should note that for $r \to 0$ i.e. near the axis of the string, the line element becomes

$$ds^2 = dt^2 - t^{[4/(p+1)]}dz_i^2 - dr^2 - a^2H^4r^2d\theta^2 \quad (17)$$

The non-static metric (17) shows that the proper volume becomes zero at $t \to 0$ and hence there is a disc like singularity in space-time.

Now we calculate curvature scalar, which gives

$$R = [\tanh(aH)r]^2\left[4aH^2 - 2a^2H^2 + 2apH^2 - \left(\frac{5}{2}\right)pH^2 - \left(\frac{3}{2}\right)H^2 - \frac{1}{2}p^2H^2\right] - 4[2aH^2 - a^2H^2 + apH^2] \quad (18)$$

which is evidently time-independent.

This result is similar to the case for non-static global string both in general relativity and in dilaton gravity and local string in Brans–Dicke theory in 4D cases (Gregory, 1996; Dando and Gregory, 1998; Sen, 1998).

It can be seen from (18) that the space-time becomes singular at different finite distances from the axis of the string for different p's i.e. $p = 1, 2, 3, \ldots,$ etc.

2. CASE – II : $b_0 \neq 0$

In this case, we solve Eqs. (10) and (11) to yield

$$e^b = \left[\cosh\left\{\frac{1}{2}E(p+1)t\right\}\right]^{[2/(p+1)]} \quad (19)$$

where $E^2 = [2b_0/p(p+1)]$

$$\int \left[De^{-2A} + \left(\frac{c}{a}\right) + \left\{\frac{d}{(a-1)}\right\} e^{-2aA} \right]^{-1/2} dA = \pm(r - r_0) \qquad (20)$$

where D and r_0 are integration constants.

From Eq. (20), one can get solution of A in closed form only for $D = 0$.

Hence we get,

$$e^{2A} = L^2 \sinh^2(Hr) \qquad (21)$$

where $L^2 = [ad/c(a-1)]$, $H^2 = (c/a)$ and r_0 is taken as zero without any loss of generality.

The expression for C is

$$C = LH \cosh(Hr) \qquad (22)$$

The string density has the following form

$$8\pi GF(r,t) = L^{-2}[\text{cosech}(Hr)]^2 \left[\Lambda + H^2 + 2pH^2 + \frac{1}{2}p(p-1)H^2 \coth^2(Hr) \right]$$

$$-\frac{1}{2}p(p-1)E^2 \tanh^2 \frac{1}{2}\{E(p+1)t\} \cdot \{\Lambda p/b_0(p+1)\} \,\text{cosech}^2(Hr)] \qquad (23)$$

In this case the element is

$$ds^2 = L^2 \sinh^2(Hr) \left[dt^2 - \left\{\cosh\left(\frac{1}{2}\right) E(p+1)t\right\}^{[4/(p+1)]} dz_i^2 \right] - dr^2$$

$$- H^2 L^2 \cosh^2(Hr) d\theta^2 \qquad (24)$$

The non-static metric (24) shows that the proper volume never vanishes for any value of t.

Thus the space-time is non-singular w.r.t. time.

This fact is reflected in curvature scalar given by

$$R = [\coth(Hr)]^2 \left[\left\{ \frac{4p\Lambda}{(p+2)} \right\} + 2pH^2 - H^2(p^2 + 3p - 2) \right]$$

$$- \left[\left\{ \frac{4p\Lambda}{(p+2)} \right\} + 2pH^2 + 2(p+2)H^2 \right] \qquad (25)$$

which is evidently time-independent.

Thus the proper volume and curvature scalar never vanish with time.

But from Eq. (25), we see that the space-time becomes singular at finite distances from the axis of the string at different points for different "p."

In conclusion, we have found a class of exact interior solutions describing local cosmic string in a higher-dimensional space. It is surprising to note that in case-I, the angular part becomes insignificant if the number of coordinates "*p*" of lower-dimensional space-time increasing indefinitely. Whereas in case-II, if *p* is large enough, all the metric coefficients shrink with "*p*." Thus if one assumes that in the early stages of the universe, the dimension is much greater than 4, then one can see, in our model, either only angular part was insignificant or entire space-time was insignificant in the early stages of the universe. Unlike the Arkani-Hamed *et al.* (1998) set up, the extra dimension here, are not at the millimeter scale but are infinite. In future, one should give a serious attention to the fact that local strings are alternative to domain walls in compactification processes.

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